Weak Maximum Principle

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We will consider the *weak maximum principle*, which states that a solution (in fact a subsolution) to an elliptic differential equation on an open set Ω attains its maximum value on the boundary of Ω.

Let Ω to be an open set in $\mathbb{R}^n$. We define the boundary of Ω to be

$$\partial\Omega = \overline{\Omega} \setminus \Omega,$$

where $\overline{\Omega}$ denotes the closure of Ω.

We will consider

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu \geq 0 \text{ in } \Omega,$$

where $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ and a^{ij}, b^i, and c are (real-valued) functions on Ω. When considering maximum principles, we have three cases depending on the sign of c to determine what type of maximum values $u(y)$ of u for $y \in \overline{\Omega}$ that we consider:

(a) When $c = 0$ on Ω, we consider the maximum value of u.

(b) When $c \leq 0$ on Ω, we consider nonnegative maximum values of u, i.e. maximum values where $u(y) \geq 0$.

(c) When we assume no sign restriction on c, we consider zero maximum values of u, i.e. maximum values where $u(y) = 0$.

Note that some lemmas and theorems consider some cases and not others.

Lemma 1 (Strict Maximum Principle)**.** *Let Ω be an open set in $\mathbb{R}^n$. Suppose $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies*

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu > 0 \text{ in } \Omega$$

for some functions a^{ij}, b^i, and c on Ω. Suppose L is an elliptic operator (i.e.

$$a^{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2 \text{ for all } x \in \Omega,\ \xi \in \mathbb{R}^n$$

for some $\lambda(x) > 0$). Then

(a) If $c = 0$ on Ω, u does not attain an interior maximum at any $y \in \Omega$.

(b) If $c \leq 0$ on Ω, u does not attain a nonnegative interior maximum at any $y \in \Omega$.

(c) *When we assume no sign restriction on c, u does not attain an interior maximum at any $y \in \Omega$ with $u(y) = 0$.*

Remark 1. *Note that in Case (a) (and similarly for Cases (b) and (c)) it is possible that u does not attain its supremum on $\overline{\Omega}$. However, if u does attain its maximum value at some point on $\overline{\Omega}$, then it follows from the Strict Maximum Principle that u attains its maximum value on $\partial\Omega$.*

Proof. Suppose $y \in \Omega$ is an interior maximum value of u. Then by the first derivative test $Du(y) = 0$ and by the second derivative test the eigenvalues of $D^2u(y)$ are nonpositive. Since $D^2u(y)$ is a symmetric matrix,

$$P^T D^2u(y) P = \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

for some $\lambda_i \leq 0$ and orthogonal $n \times n$ matrix P. Let $A = (a^{ij}(y))$ and $\widetilde{A} = (\widetilde{a}^{ij}) = P^T AP$ as $n \times n$ matrices. Then by the ellipticity of L, for all $\xi \in \mathbb{R}^n$ and $\zeta = P\xi$,

$$\sum_{i=1}^n \widetilde{a}^{ij}\xi_i\xi_j = \text{trace}(\xi^T \widetilde{A}\xi) = \text{trace}(\xi^T P^T AP\xi) = \text{trace}((P\xi)^T AP\xi)$$

$$= \text{trace}(\zeta^T A\zeta) = \sum_{i=1}^n a^{ij}\zeta_i\zeta_j > 0.$$

In particular by letting ξ being the i-th coordinate vector in $\mathbb{R}^n$, $\widetilde{a}^{ii} > 0$. Also,

$$\sum_{i,j=1}^n a^{ij}(y)D_{ij}u(y) = \text{trace}(AD^2u) = \text{trace}(AP\Lambda P^T) = \text{trace}(P^T AP\Lambda P^T P)$$

$$= \text{trace}(P^T AP\Lambda) = \text{trace}(\widetilde{A}\Lambda) = \sum_{i=1}^n \widetilde{a}^{ii}\lambda_i,$$

where we use the fact that $\text{trace}(B) = \text{trace}(P^T BP)$ for any $n \times n$ matrix B since P is orthogonal, so

$$Lu(y) = \sum_{i=1}^n \widetilde{a}^{ii}\lambda_i + c(y)u(y) > 0.$$

But $\widetilde{a}^{ii} > 0$ for all $i = 1, 2, \ldots, n$, $\lambda_i \leq 0$ for all $i = 1, 2, \ldots, n$, and $c(y)u(y) \leq 0$, giving us a contradiction. Therefore no such interior maximum y exist. □

Theorem 1 (Weak Maximum Principle)**.** *Let Ω be a bounded open set in $\mathbb{R}^n$. Suppose $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies*

$$Lu = a^{ij}D_{ij}u + b^iD_iu + cu \geq 0 \text{ in } \Omega$$

for some functions a^{ij}, b^i, and c on Ω. Suppose L is an elliptic operator and

$$\sup_\Omega \frac{|b^i|}{\lambda} < \infty.$$

Then:

(a) If $c = 0$ on Ω,

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

(b) If $c \leq 0$ on Ω,

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+,$$

where $u^+(x) = \max\{u(x), 0\}$ at each $x \in \Omega$.

Remark 2. *Note that in Case (a) (and similarly for Case (b)) the Weak Maximum Principle only shows that u attains its maximum value on $\partial\Omega$. It is possible that u attains its maximum value on both the interior and the boundary of Ω.*

Example: Note that the Weak Maximum Principle does not generally hold if $c(x) > 0$ for some $x \in \Omega$. For example, consider

$$L = \frac{\partial^2}{\partial x^2} + \ell^2, \quad u(x) = \sin(\ell x), \quad x \in [0, 2\pi/\ell].$$

Then $Lu = 0$ in $(0, 2\pi/\ell)$ but u attains a positive interior maximum value of 1 at $x = \pi/2\ell$ while $u(x) = 0$ at the endpoints $x = 0$ and $x = 2\pi/\ell$.

Proof of Case (a). Suppose without loss of generality that Ω is contained in a slab $\{(x_1, x_2, \ldots, x_n) : 0 < x_1 < d\}$ for $d > 0$. Let

$$\beta = \sup_{\Omega} \frac{|\beta|}{\lambda}.$$

Let

$$v(x) = e^{\alpha x_1} \text{ for } x \in \Omega$$

for some constant $\alpha > 0$ to be determined and consider

$$w_\varepsilon(x) = u(x) + \varepsilon v(x) \text{ for } x \in \Omega$$

for $\varepsilon > 0$ arbitrary. We compute that

$$\begin{aligned} Lv &= \alpha^2 a^{11} e^{\alpha x_1} + \alpha b_1 e^{\alpha x_1} \\ &\geq (\alpha^2 \lambda - \alpha \lambda \beta) e^{\alpha x_1} \\ &> 0 \end{aligned}$$

provided α is chosen large enough that $\alpha > \beta$. (Note that conceptually what this computation shows is that the $a^{ij} D_{ij} v$ terms dominate the $b^i D_i v$ and cv terms. The idea that the $a^{ij} D_{ij}$ is the dominant term in L will appear in many proofs in this course.) By linearity,

$$Lw_\varepsilon = Lu + \varepsilon Lv > 0 \text{ in } \Omega$$

for all $\varepsilon > 0$. Since Ω is bounded, w_ε attains its maximum value somewhere on $\overline{\Omega}$. By the Strict Maximum Principle, w_ε attains its maximum value on $\partial\Omega$ and thus

$$\sup_{\Omega} w_\varepsilon = \sup_{\partial\Omega} w_\varepsilon.$$

Since $w_\varepsilon = u + \varepsilon v$ and $1 \leq v \leq e^{\alpha d}$ on Ω,

$$\sup_{\Omega} u + \varepsilon \leq \sup_{\partial\Omega} u + \varepsilon e^{\alpha d}.$$

Letting $\varepsilon \downarrow 0$,

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u.$$

□

Proof of Case (b). Let $\Omega_+ = \{x \in \Omega : u(x) > 0\}$ and $L_0 = a^{ij} D_{ij} + b^i D_i$. Note that if $\Omega_+ = \emptyset$, then the conclusion of Case (b) is trivially true, so we may suppose $\Omega_+ \neq \emptyset$. Since $L_0 u = Lu - cu \geq 0$ on Ω_+, by Case (a),

$$\sup_{\Omega_+} u = \sup_{\partial\Omega_+} u. \tag{1}$$

Observe that

$$\partial\Omega_+ \subseteq \{x \in \partial\Omega : u(x) > 0\} \cup \{x \in \Omega : u(x) = 0\}.$$

If $u = 0$ on $\partial\Omega_+$,

$$0 < \sup_{\Omega_+} u = \sup_{\partial\Omega_+} u = 0,$$

yielding a contradiction, so there exists points $x \in \partial\Omega_+$ such that $x \in \partial\Omega$ and $u(x) > 0$ and consequently

$$\sup_{\partial\Omega_+} u = \sup_{\partial\Omega} u^+. \tag{2}$$

Now by (1) and (2),

$$\sup_{\Omega} u = \sup_{\Omega_+} u = \sup_{\partial\Omega_+} u \leq \sup_{\partial\Omega} u^+.$$

□

There are several important consequences of the weak maximum.

Corollary 1 (Uniqueness of Solutions to the Dirichlet Problem)**.** *Let Ω be a bounded open set in $\mathbb{R}^n$. Consider the Dirichlet problem*

$$\begin{aligned} Lu = a^{ij} D_{ij} u + b^i D_i u + cu = f \text{ in } \Omega, \\ u = \varphi \text{ on } \partial\Omega, \end{aligned}$$

for some functions a^{ij}, b^i, c, and f on Ω and $\varphi \in C^0(\partial\Omega)$ such that L is an elliptic operator,

$$\sup_{\Omega} \frac{|b^i|}{\lambda} < \infty,$$

and $c \leq 0$ in Ω. Then there is at most one solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ to the Dirichlet problem (i.e. there may be no solution or a unique solution but there cannot be two or more solutions).

Proof. Suppose u_1 and u_2 are two solutions to the Dirichlet problem. Then

$$\begin{aligned} L(u_1 - u_2) = 0 \text{ in } \Omega, \\ u_1 - u_2 = 0 \text{ on } \partial\Omega. \end{aligned}$$

By the Weak Maximum Principle, $u_1 - u_2 \leq 0$ on $\overline{\Omega}$. By swapping u_1 and u_2, $u_2 - u_1 \leq 0$ on $\overline{\Omega}$. Therefore $u_1 - u_2 = 0$ on $\overline{\Omega}$, i.e. $u_1 = u_2$ on $\overline{\Omega}$. □

Corollary 2 (Comparison Principle). *Let Ω be a bounded open set in $\mathbb{R}^n$. Let $L = a^{ij} D_{ij} + b^i D_i + c$ be an elliptic operator for some functions a^{ij}, b^i, and c on Ω such that*

$$\sup_{\Omega} \frac{|b^i|}{\lambda} < \infty$$

and $c \leq 0$ in Ω. If $u, v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that $Lu \geq Lv$ in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Proof. Observe that

$$\begin{aligned} L(u - v) &\geq 0 \text{ in } \Omega, \\ u - v &\leq 0 \text{ on } \partial\Omega. \end{aligned}$$

By the Weak Maximum Principle, $u - v \leq 0$ on $\partial\Omega$, i.e. $u \leq v$ on $\overline{\Omega}$. □

Note that we have the following consequence of Corollary 2. Let Ω be a bounded open set in $\mathbb{R}^n$, $L = a^{ij} D_{ij} + b^i D_i + c$ be an elliptic operator for some functions a^{ij}, b^i, and c on Ω such that

$$\sup_{\Omega} \frac{|b^i|}{\lambda} < \infty$$

and $c \leq 0$ in Ω, and $f : \Omega \to \mathbb{R}$. Consider the equation $Lu = f$ in Ω. Given $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, we say

(1) u is a *solution* if $Lu = f$ in Ω,

(2) u is a *subsolution* if $Lu \geq f$ in Ω, and

(3) u is a *supersolution* if $Lu \leq f$ in Ω.

Corollary 2 has the following obvious consequence. Suppose $u_{\text{sub}}, u_{\text{soln}}, u_{\text{super}} \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that u_{sub} is a subsolution, u_{soln} is a solution, and u_{super} is a supersolution. Hence

$$Lu_{\text{sub}} \geq Lu_{\text{soln}} \geq Lu_{\text{super}} \text{ in } \Omega.$$

Further suppose that

$$u_{\text{sub}} \leq u_{\text{soln}} \leq u_{\text{super}} \text{ on } \partial\Omega.$$

Then

$$u_{\text{sub}} \leq u_{\text{soln}} \leq u_{\text{super}} \text{ on } \overline{\Omega}.$$

References: Gilbarg and Trudinger, Section 3.1.